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Total Stability of Perturbed Systems of Differential Equations

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Abstract—The notion of Lipschitz stability for systems of ordinary differential equations (ODE) was introduced. In this paper, we will extend the total stability notion to a new type of stability called total Lipschitz stability. Some criteria and results are given. Our technique depends on Liapunov's direct method. MSC-34D20.

Keywords—Uniform stability, Uniform T_i-total stability, $i = 1, 2$, Uniform Lipschitz stability, Uniform total Lipschitz stability.

1. INTRODUCTION

Consider the systems

$$x' = f(t, x), \tag{1.1}$$

and the perturbed system

$$x' = f(t, x) + h(t, x), \tag{1.2}$$

where $f, h \in C[J \times R^n, R]$, $J = [t_0, \infty)$, $f(t, 0) = h(t, 0) = 0$, with $x(t_0, t_0, x_0) = x_0$, R^n is the n-dimensional Euclidean real space, $\|x\|$ is any norm of the vector $x \in R^n$, and $R = (\infty, -\infty)$. Define $S_\rho = \{x, x \in R^n, \|x\| < \rho, \rho > 0\}$.

The variety of problems of the qualitative properties of differential equations has been successful in different approaches based on Liapunov's direct method, for ordinary and functional differential equations. There are many results concerned with relationships between the total stability and uniform asymptotic stability (see [1–5]).

For a system of differential equations, uniformly asymptotically stable implies totally stable. It is known that the converse is not generally true. In [6] Dannan and Elaydi introduced the notion of Lipschitz stability for systems (1.1) and (1.2). Furthermore, they investigated and improved the relation between Lipschitz stability and Liapunov stability (see [7]). Many authors discussed and proved the necessary and sufficient condition for the zero solution of systems of differential equations notions to be T_1 and T_2-total stable.

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The aim of the paper is to extend both the notions Lipschitz stability of [7] and T_i-stability of [8], $i = 1, 2$ for system (1.1) to the so-called T_i-total Lipschitz stability, $i = 1, 2$. Furthermore, we prove that uniform asymptotic stability implies uniform Lipschitz stability of the zero solution of (1.2) which is inconsistent with the conjecture of Dannan and Elaydi [7].

Consider the comparison system

$$u' = G(t, u), u(t_0) \tag{1.3}$$

and the perturbed system

$$u' = G(t, u) + p(t), \tag{1.4}$$

where $G \in C[J \times R^n, R]$, $G(t, 0) = 0$, and $p(t) \in [J \times R^+]$. Now, as in [5,8], we define a Liapunov function $V(t, x) \in C[J \times R^n, R]$, and the function

$$D^+V(t) = \limsup_{\delta \to 0} \frac{1}{\delta} [V(t + \delta, x + \delta f(t, x)) - V(t, x)]. \tag{1.5}$$

The following definitions will be needed.

DEFINITION 1.1. *The zero solution of system (1.1) is said to be T_1- totally Lipschitz stable if for $\epsilon > 0$, there exist positive numbers $\delta_1 > 0$, $\delta_2 > 0$, and $M > 1$ such that for a solution $x(t, t_0, x_0)$ of the perturbed system (1.2), the inequality*

$$\|x(t, t_0, x_0)\| \le M \|x_0\|$$

holds, provided that

$$\|x_0\| \le \delta_1, \quad \|h(t, x)\| \le \delta_2, \qquad \text{for } \|x\| < \epsilon, \quad t \in J.$$

DEFINITION 1.2. *The zero solution of system (1.1) is said to be T_2- totally Lipschitz stable (or totally Lipschitz stable under permanent perturbations bound in the mean) if for $\epsilon > 0$, there exist positive numbers $\delta_1(\epsilon) > 0$, $\delta_2(\epsilon) > 0$, and $M > 1$ such that for the perturbed system (1.2), the inequality*

$$\|x(t, t_0, x_0)\| \le M \|x_0\| \tag{1.6}$$

holds, provided that

$$\|x_0\| \le \delta_1, \quad \|h(t, x)\| \le \lambda(t), \qquad \text{for } \|x\| < \epsilon,$$

and *(1.7)*

$$\int_{t_0}^{t_0+T} \lambda(s)\, ds \le \delta_2, \qquad T > 0.$$

Any T_i-totally Lipschitz stability, $i = 1, 2$ can be similarly defined.

2. LIPSCHITZ STABILITY

In this section, we will discuss uniform asymptotic stability implies uniform Lipschitz stability of (1.1) which is inconsistent with the conjecture of Dannan and Elaydi [7].

LEMMA 2.1. *If the zero solution of (1.1) is uniformly asymptotically stable, then there exists a continuous function $V(t, x) \in [R^+ \times R^+, R]$ such that*

$$\|x\| \le V(t, x) \le M\|x\|, \qquad M > 1,$$

for any solution $x(t)$ of (1.1).

PROOF. Since the zero solution of (1.1) is uniformly asymptotically stable, by Theorem 5.4.4 of [9], we get

$$\|x\| \le c\|x(t_1)\|\sigma(t - t_1), \qquad t \ge t_1, \quad c \in \kappa, \quad \sigma \in \iota. \tag{2.1}$$

Thus, choosing σ and c as $\sigma(t) = e^{-\alpha(t-t_1)}$, $\alpha \geq 0$ and

$$c\|x\| = M\|x(t_1)\|, \qquad M > 1.$$

Consequently, inequality (2.1) becomes

$$\|x\| = M\|x(t)\|e^{-\alpha(t-t_1)}, \qquad t \geq t_1. \tag{2.2}$$

Now, if the scalar function $V(t,x)$ is chosen as

$$V(t,x) = \sup_{t \geq t_1} \|x\|. \tag{2.3}$$

Hence, we get

$$\|x\| \leq V(t,x) \leq M\|x\|.$$

The proof is completed.

THEOREM 2.2. *Let the hypotheses of Lemma 2.1 be satisfied and $f(t,x)$ in (1.1) be locally Lipschitzian in t. Then the zero solution of (1.1) is uniformly Lipschitz stable.*

PROOF. From Lemma 2.1, the zero solution of (1.1) is uniformly asymptotically stable, by using Theorem 3.6.9 of [9]. There exists a continuous function $V(t,x) \in [R^+ \times R^+, R]$ satisfying

(1) $\|x\| \leq V(t,x) \leq M\|x\|$, $M > 1$;
(2) $\|V(t,x) - V(t,y)\| \leq \|x - y\|$, for $(t,x),(t,y) \in J \times R^n$;
(3) $V'(t,x)_{(1.1)} \leq -c\|x\|$, $c \in \mathcal{K}$.

Thus, it clear that condition (6) implies $V'(t,x) \leq 0$, and therefore, the conditions of Theorem 1.2 of [7] are satisfied. Hence, the zero solutions of (1.1) are uniformly Lipschitz stable. The proof is completed.

3. TOTAL LIPSCHITZ STABILITY

In this section, we discuss the notion of T_i-total Lipschitz stability of (1.1) which connects between both notions of Lipschitz stability of [6,7], and T_i-total stability of [8], $i = 1, 2$.

The following definitions will be needed in this section.

THEOREM 3.1. *Suppose that $f(t,x)$ in (1.1) is locally Lipschitzian in x uniformly in t, and $h(t,x)$ is a bounded function. Then the zero solution of (1.1) is uniformly T_1-totally Lipschitz stable iff there exists a continuous function $V(t,x)$ for $t \geq t_0$, and $\|x\| < \delta$, such that*

(4) $\|x\| \leq V(t,x) \leq L\|x\|$, $L > 1$;
(5) $\|V(t,x) - V(t,y)\| \leq L^*\|x - y\|$, *for* $t \geq t_0$, $L^* > 1$;
(6) $V'(t,x) \leq 0$.

PROOF. Let the zero solution of (1.1) be uniformly T_1-totally Lipschitz stable. Then there exist $M > 1$, $\delta_1 > 0$, and $\delta_2 > 0$ such that

$$\|x(t,t_0,x_0)\| \leq M\|x_0\|, \tag{3.1}$$

whenever $\|x_0\| \leq \delta_1$, $\|h(t,x)\| \leq \delta_2$, $t \geq t_0$.

Following [7], we choose the function

$$V(t,x) = \sup_{s \geq 0} \|x(t+s,t,x)\| \left(e^{-s-t} + 1\right). \tag{3.2}$$

Then

$$\begin{aligned}\|x(t,t,x)\| \leq \|x(t,t,x)\| \left(1 + e^{-1}\right) &\leq V(t,x) \\ &\leq M \sup_{s \geq 0} \|x(t,t,x)\| \left(e^{-s-t} + 1\right) \\ &\leq 2M\|x\| \leq L\|x\|.\end{aligned}$$

This proves (4).

Since $f(t,x)$ in (1.1) is locally Lipschitzian in x uniformly in t, there is $q = q\ (M,\delta)$ such that

$$\|x(t+s,t,x)\| - \|x(t+s,t,y)\| \leq e^{qs}\|x-y\|,$$

for

$$\|x\| \leq \delta, \quad \|y\| \leq \delta, \qquad \delta \geq 0.$$

We note that

$$\|x(t+s,t,x)\| \leq M\|x\| \leq M\delta \qquad \text{and} \qquad \|y(t+s,t,y)\| \leq M\|y\| \leq M\delta.$$

Let $N > 0$ be a constant such that $M = e^N$. Then

$$\sup_{s\geq 0} \|x(t+s,t,x)\| \left(e^{-s-t}+1\right) \leq \sup_{s\geq 0} \|x\| \left(e^N + e^{N-s-t}\right).$$

Hence, the above sup is realized for $0 \leq t+s \leq N$ in case $t > N$ and if $s = 0$ in case $t \geq N$. Thus, for $\|x\| \leq \delta$, and $\|y\| \leq \delta$,

$$\begin{aligned}
\|V(t,x) - V(t,y)\| &\leq \sup_{s\geq 0}(\|x(t+s,t,x)\| - \|x(t+s,t,y)\|) \left(e^{-s-t}+1\right) \\
&\qquad (0 \leq t+s \leq N \text{ if } t < N, \text{ and } s = 0 \text{ if } t \geq N) \\
&\leq \sup_{s\geq 0} e^q \|x-y\| \left(e^{-s-t}+1\right) \\
&\leq L\|x-y\|, \qquad L > 1.
\end{aligned}$$

This proves (5).

The proof of (6) follows from (3.2). We get

$$\begin{aligned}
V' &= \lim_{\delta\to 0^+} \frac{1}{\delta} \left[V(t+\delta, x(t+\delta,t,x)) - V(t,x)\right] \\
&= \lim_{\delta\to 0^+} \frac{1}{\delta} \left[\sup_{s\geq\delta} \|x(t+\delta+s,t,x)\| \left(e^{-s-t+h}+1\right)\right. \\
&\quad \left. - \sup_{s\geq\delta} \|x(t+s,t,x)\| \left(e^{-s-t}+1\right)\right] \leq 0.
\end{aligned}$$

Since

$$\begin{aligned}
\|V(t+\delta,x+y) - V(t,y)\| \leq{}& \|V(t+\delta,x+y) - V(t+\delta, x(t+\delta,t,x+y))\| \\
&- \|V(t+\delta, x(t+\delta,t,x+y)) - V(t,x,y)\| \\
&+ \|V(t,x+y) - V(t,y)\|.
\end{aligned}$$

From the continuity of solution (1.1) and the Lipschitzian of $V(t,x)$, the first and the third terms can be made small, and the second term can be seen to be small if δ is small by an argument similar to that used to prove that $V'(t,x)$ exists.

Conversely, let conditions (4)–(6) be satisfied and from the assumption that $h(t,x)$ is a bounded function on its argument, then for any solution $x(t,t_0,x_0)$ of system (1.2) and $M > 1$,

$$\|x(t,t_0,x_0)\| \leq V(t,x) \leq V(t_0,x_0) \leq M\|x_0\|,$$

for $\|x_0\| \leq \delta_1$, and $\|h(t,x)\| \leq \delta_2$, then the zero solution is uniformly T_1-totally Lipschitz stable, and the proof is completed.

REMARK. We can see that if the zero solution of (1.1) is uniformly T_1- totally asymptotically stable, then there exists a continuous function $V(t,x) \in C[R^+ \times R^+, R]$ such that

$$\|x\| \leq V(t,x) \leq M\|x\|, \qquad M > 1,$$

for any solution $x(t)$ of (1.1).

THEOREM 3.2. *Let there exist two functions $f(t,x)$ that is defined in (1.1) and $V(t,x) \in C[J \times R^n, R]$. Suppose the zero solution of (1.1) is uniformly asymptotically stable. Then it is uniformly T_1-totally Lipschitz stable.*

PROOF. Since the zero solution of (1.1) is uniformly asymptotically stable, it follows from Theorem 3.6.9 of [8], there exist two functions $f(t,x) \in C[J \times R^n, R]$, $f(t,0) = 0$ and $V(t,x) \in C[J \times R^n, R]$, $V(t,0) = 0$ satisfying the following conditions.

(7) $\|f(t,x) - f(t,y)\| \leq L(t)\|x - y\|$, and

$$\left| \int_{t_0}^{t_0+T} L(s)\,ds \right| \leq N|T|, \qquad \text{for } (t,x),(t,y) \in J \times S_\rho, \quad \text{and } T > 0.$$

(8) $b\|x\| \leq V(t,x) \leq a(\|x\|)$, $a, b \in \mathcal{K}$.

(9) $\|V(t,x) - V(t,y)\| \leq M\|x - y\|$, for $(t,x),(t,y) \in J \times R^n$.

(10) $D^+V(t,x) \leq -c|V(t,x)|$, $c \in \mathcal{K}$.

Let $0 < \epsilon < \delta(\epsilon)$ be given. Choose $\delta_1 = \delta_1(\epsilon)$ such that

$$b(\epsilon) > \delta_1. \tag{3.3}$$

Define $m(t) = V(t,x)$, where $x(t) = x(t,t_0,x_0)$ is a solution (1.2) such that $\|x_0\| \leq \delta_1$. From condition (8), we have

$$m(t_0) = V(t_0,x_0) \leq a\|x_0\| \leq a(\delta_1).$$

Now, we assume that

$$m(t_0) < b(M\|x_0\|), \qquad t \leq t_0, \quad M \geq 1. \tag{3.4}$$

If this is false, there exist two numbers $t_1 > t_2 > t_0$ such that

$$m(t_2) = a(\delta_1), \quad m(t_1) = b(\epsilon), \qquad \text{and } m(t) \geq a(\delta_1), \quad t_2 \leq t \geq t_1.$$

Thus,

$$D^+m(t_2) = D^+(a(\delta_1)) \geq 0. \tag{3.5}$$

As in [8], we get

$$D^+m(t_2) < -c[a(\delta_1)] + M(\delta_2) = 0,$$

which contradicts (3.5). Therefore, the zero solution of (1.1) is uniformly T_1-totally Lipschitz stable.

4. CONCLUSION

The reader can check Figure 1, which appears as relations between different types of stability.

$$\begin{tikzcd}
UT_1LS \arrow[r] \arrow[d] & ULS \arrow[d] & UT_2LS \arrow[l] \arrow[d] \\
UT_1S \arrow[r] \arrow[dr] & US \arrow[d] & UT_2S \arrow[l] \arrow[dl] \\
 & S &
\end{tikzcd}$$

Figure 1.

UT_iLS : Uniformly T_i-totally Lipschitz stable, $i = 1, 2$.

UT_iS : Uniformly T_i-totally stable, $i = 1, 2$ [8].

ULS : Uniformly Lipschitz stable [7].

US : Uniformly stable [8].

S : Stable [8].

REFERENCES

1. A. Halanay, *Differential Equations, Stability, Oscillations*, Time Lage, Academic Press, (1969).
2. Y. Hino, Total stability and uniform asymptotic stability for linear functional differential equations, *Funke. Ekvac.* **24**, 345–349, (1981).
3. J. Kato and T. Yoshizawa, A relationship between uniform asymptotic stability and total stability, *Funke. Ekvac.* **12**, 233–238, (1969).
4. J. Kato, Uniform asymptotic stability and total stability, *Tohoku Math. J.* **22**, 254–296, (1970).
5. T. Yoshizawa, *Stability Theory by Liapunov's Second Method*, Math. Soc. Japan, Tokyo, (1966).
6. F. Dannan and S. Elaydi, Lipschitz stability of nonlinear systems of differential equations, *J. Math. Anal. Appl.* **113**, 562–577, (1986).
7. F.M. Dannan and S. Elaydi, Lipschitz stability of nonlinear systems of differential equations II Liapunov function, *J. Math. Anal. Appl.* **143**, 517–529, (1989).
8. V. Lakshmikantham and S. Leela, *Differential and Integral Inequalities, Volume 1*, Academic Press, New York, (1969).
9. M. Rama and M. Rao, *Ordinary Differential Equations Theory and Applications*, New York, (1980).
10. S. Rajalaksmy and S. Sivasundaram, Vector Liapunov functions and the technique in perturbation theory, *J. Math. Anal. Appl.* **143**, 517–529, (1992).